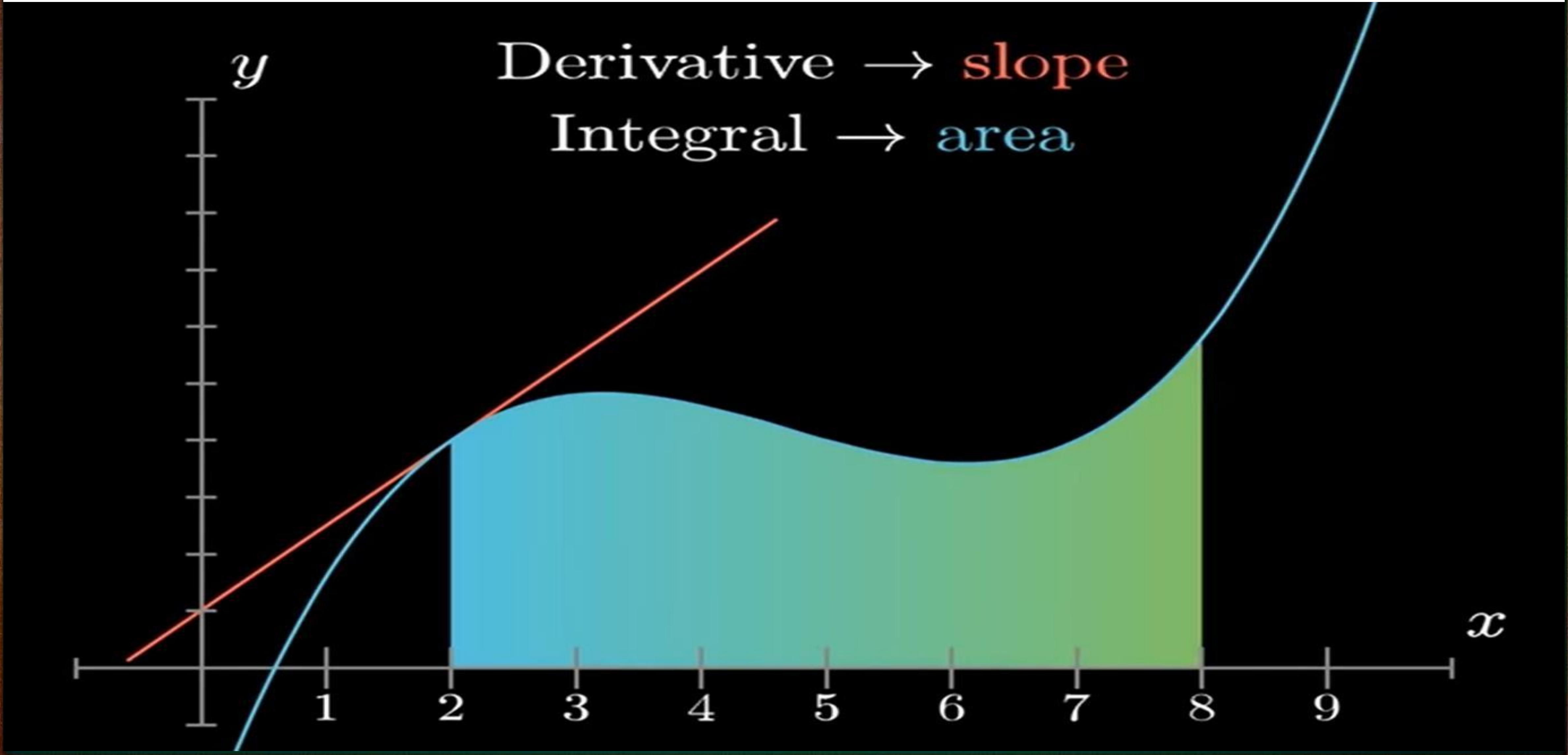
# TWO MAIN PROBLEMS OF CALCULUS



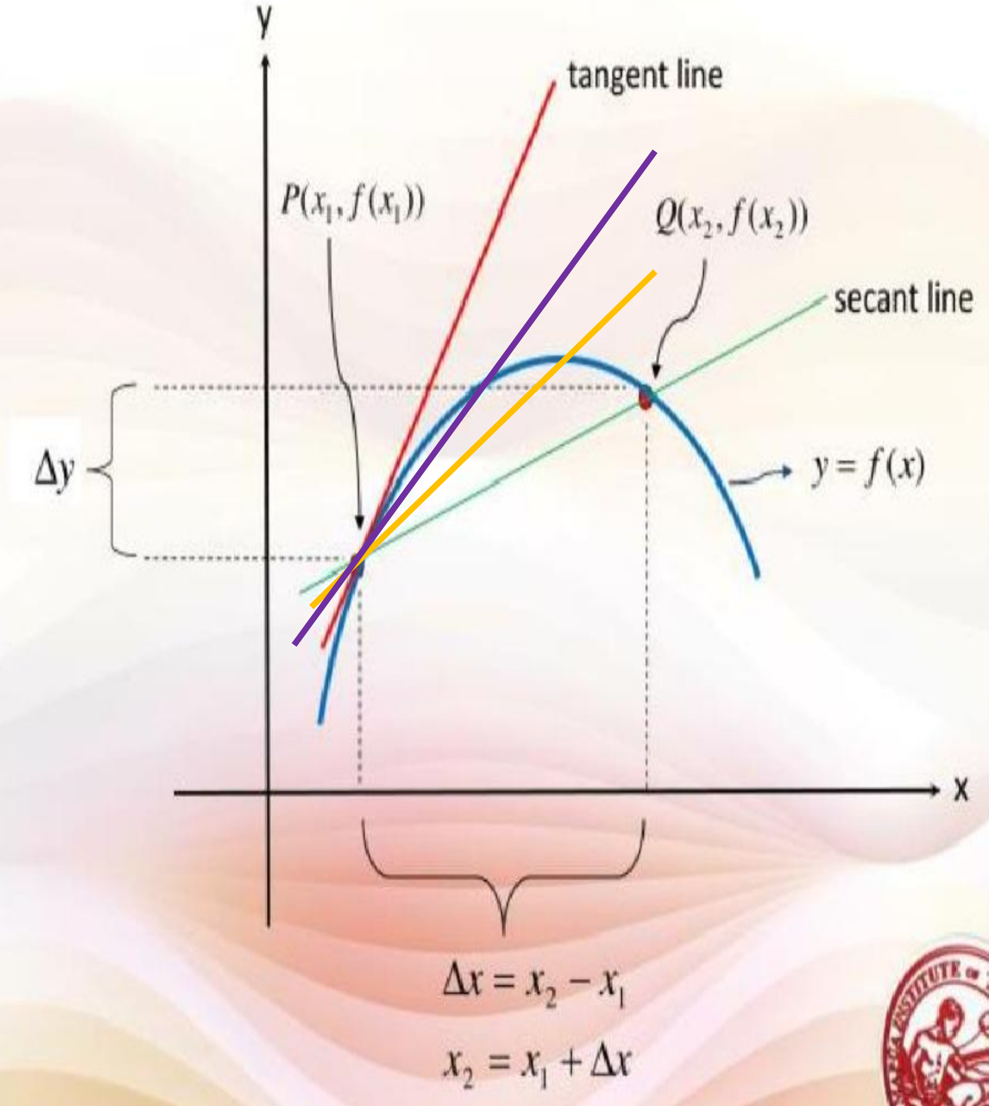
# 3.1 Derivative of a Function

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ is called the derivative of $f$ at $a$.

We write: $$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

"The derivative of $f$ with respect to $x$ is ..."

There are many ways to write the derivative of $y = f(x)$

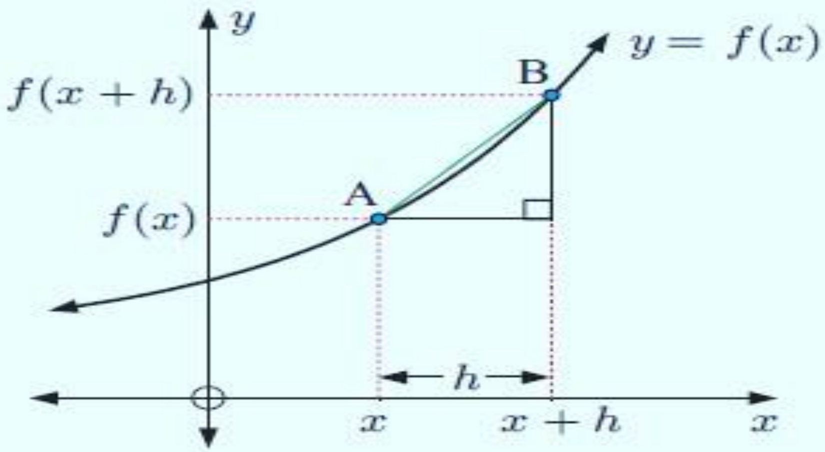


# Differentiability

- Differentiability- being able to find an existing derivative for a given point on a function
- For a function to be differentiable at a specific point, it needs to be continuous at that point and the derivative must also be continuous at that point
- Places at which functions are non-differentiable:
    - Cusps
    - Vertex
    - Vertical Asymptote
    - Jump
    - Hole
    - Vertical tangent line

# DIFFERENTIATION FROM FIRST PRINCIPLES

Consider a general function $y = f(x)$ where A is the point $(x, f(x))$ and B is the point $(x + h, f(x + h))$.




The chord [AB] has gradient $= \dfrac{f(x+h) - f(x)}{x + h - x}$

$= \dfrac{f(x+h) - f(x)}{h}$

If we let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is the limiting value of $\dfrac{f(x+h) - f(x)}{h}$

as $h$ approaches 0, or $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

This formula gives the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$, for any value of the variable $x$ for which this limit exists. Since there is at most one value of the gradient for each value of $x$, the formula is actually a function.

The **derivative function** or simply **derivative** of $y = f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

## ALTERNATIVE NOTATION

If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

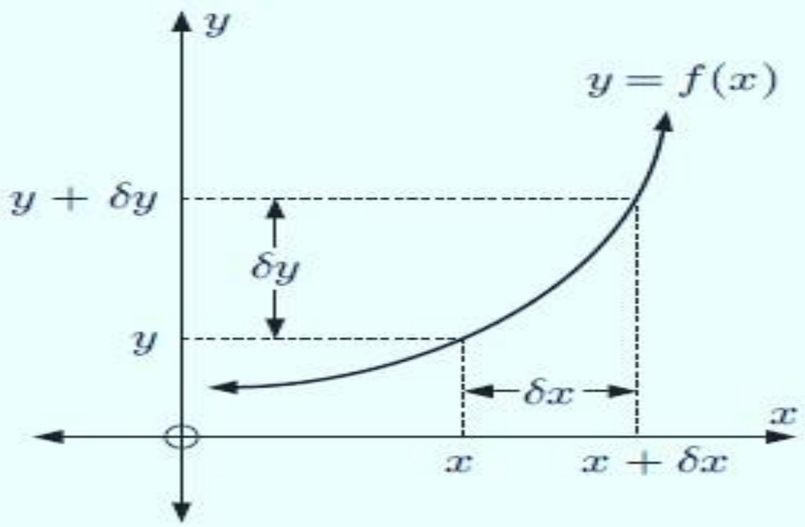
If we are given $y$ in terms of $x$ then $y'$ or $\frac{dy}{dx}$ are commonly used to represent the derivative.

$\frac{dy}{dx}$ reads "dee $y$ by dee $x$" or "the derivative of $y$ with respect to $x$".

$\frac{dy}{dx}$ is **not a fraction**. However, the notation $\frac{dy}{dx}$ is a result of taking the limit of a fraction. If we replace $h$ by $\delta x$ and $f(x+h) - f(x)$ by $\delta y$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{becomes}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}.$$




## THE DERIVATIVE WHEN $x = a$

The gradient of the tangent to $y = f(x)$ at the point where $x = a$ is denoted $f'(a)$, where

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

# Differentiability or Derivability

A real function $f$ is said to be **derivable** or **differentiable** at a point $c$ in its domain, if its **left hand** and **right hand** derivatives at $c$ exist (i.e. finite and unique) and are equal, i.e. $Lf'(c) = Rf'(c)$. Here, at $x = c$ left hand derivative,

$$\text{LHD} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = Lf'(c)$$

and right hand derivative,

$$\text{RHD} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = Rf'(c).$$

The common value of $Lf'(c)$ and $Rf'(c)$ is known as the derivative of $f(x)$ at $x = c$ and denoted by $f'(c)$.

Also, a function is said to be differentiable in an interval $(a, b)$, if it is differentiable at every point of $(a, b)$.

A function is said to be differentiable in an interval $[a, b]$, if it is differentiable at every point of $[a, b]$ (same as continuity we take right hand derivative and left hand derivative at $a$ and $b$, respectively).

# Method to Show Differentiability of a Function

Suppose a function $f(x)$ define in a domain is given to us and we have to check its differentiability at point $x = c$ in its domain. Then, we use the following steps:

I. Firstly, write the given function say $f(x)$ and the point say $x = c$ at which we have to check differentiability.

II. Find left hand derivative (LHD) at $x = c$ by using the formula, $\text{LHD} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$.

III. Find right hand derivative (RHD) at $x = c$ by using the formula

$$\text{RHD} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

IV. If LHD = RHD at $x = c$, then $f(x)$ is differentiable at $x = c$, otherwise $f(x)$ is not differentiable at $x = c$.

**Example 1. Is f (x) = |x – 1| + |x – 2| differentiable at x = 2 ?**

**Solution.** We have : $f(x) = |x-1| + |x-2|$.

$$\therefore \quad Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \frac{[\,|(2-h)-1| + |(2-h)-2|\,] - [\,|2-1| + |2-2|\,]}{-h}$$

$$= \lim_{h \to 0} \frac{|1-h| + |-h| - 1 - 0}{-h}$$

$$= \lim_{h \to 0} \frac{1-h+h-1}{-h}$$

$$= \lim_{h \to 0} \frac{0}{-h} = \lim_{h \to 0} 0 = 0$$

and $$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[\,|(2+h)-1| + |(2+h)-2|\,] - [\,|2-1| + |2-2|\,]}{h}$$

$$= \lim_{h \to 0} \frac{1+h+h-1-0}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2.$$

Thus $Lf'(2) \neq Rf'(2)$. $[\because 0 \neq 2]$

Hence, '$f$' is not differentiable at $x = 2$.

**Example 2.** Find '$a$' and '$b$', if the function given by : $f(x) = \begin{cases} ax^2 + b, & \text{if } x < 1 \\ 2x + 1, & \text{if } x \geq 1 \end{cases}$ is differentiable at $x = 1$.

**Solution.** Since '$f$' is derivable at $x = 1$,

$\therefore$ '$f$' is continuous at $x = 1$

$$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x \to 1^-} (ax^2 + b) = \lim_{x \to 1^+} (2x + 1) = 2(1) + 1 = 3$$

$$\Rightarrow \lim_{h \to 0} (a(1-h)^2 + b) = \lim_{h \to 0} 2(1+h) + 1 = 3$$

$$\Rightarrow \quad a + b = 2 + 1 = 3$$

$$\Rightarrow \quad a + b = 3$$

$$\Rightarrow \quad a + b = 3 \quad \ldots(1)$$

Again since '$f$' is differentable at $x = 1$,

$$\therefore \quad Lf'(1) = Rf'(1)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{[a(1-h)^2 + b] - 3}{-h} = \lim_{h \to 0} \frac{[2(1+h) + 1] - 3}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{(a+b) - 2ah + ah^2 - 3}{-h} = \lim_{h \to 0} \frac{(2h+3) - 3}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{3 - ah(2-h) - 3}{-h} = \lim_{h \to 0} \frac{2h}{h} \quad [Using\ (1)]$$

$$\Rightarrow \lim_{h \to 0} \frac{-a(2-h)}{-1} = \lim_{h \to 0} \frac{2h}{h}$$

$$\Rightarrow \lim_{h \to 0} a(2-h) = \lim_{h \to 0} (2)$$

$$\Rightarrow \quad 2a = 2 \Rightarrow a = 1.$$

Putting in (1), $1 + b = 3 \Rightarrow b = 3 - 1 \Rightarrow b = 2.$

Hence, $a = 1$ and $b = 2.$

1. *Examine the differentiability of the function*

$$f(x) = \begin{cases} x[x] & , \text{if } 0 \le x < 2 \\ (x-1)x & , \text{if } 2 \le x < 3 \end{cases}$$

**NCERT Exemplar**

***Sol.*** Given, $f(x) = \begin{cases} x[x] & , \text{if } 0 \le x < 2 \\ (x-1)x & , \text{if } 2 \le x < 3 \end{cases}$

and at point $x = 2$, we have to check differentiability.

At $x = 2$, LHD $= \lim_{h \to 0} \dfrac{f(2-h) - f(2)}{-h}$

$$= \lim_{h \to 0} \frac{(2-h)[2-h] - 2[2]}{-h}$$

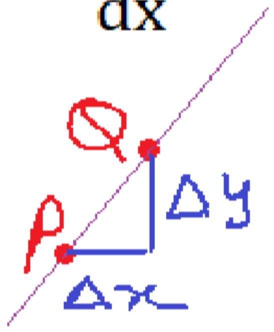
$$= \lim_{h \to 0} \frac{(2-h)(1) - 4}{-h} \qquad [\because [2-h] = 1 \text{ and } [2] = 2]$$

$$= \lim_{h \to 0} \frac{2-h-4}{-h} = \lim_{h \to 0}\left(\frac{-2-h}{-h}\right) = \lim_{h \to 0}\left(\frac{2+h}{h}\right)$$

$=$ not defined

Hence, $f(x)$ is not differentiable at $x = 2$.

# Definition of Derivative: $f'(x) = \dfrac{dy}{dx}$




1. The Derivative is the exact rate at which one quantity changes with respect to another.
2. Geometrically, the derivative is the slope of curve at the point on the curve.
3. The derivative is often called the "instantaneous " rate of change.
4. The derivative of a function represents an infinitely small change the function with respect to one of its variables.

$\dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx}$ as $\Delta x \to 0$

$\Rightarrow |PQ| \to 0$

# Algebra of Differentiation 1

- Sum/Difference $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
- Product $\dfrac{d}{dx}(uv) = v \cdot \dfrac{du}{dx} + u \cdot \dfrac{dv}{dx}$
- Quotient $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

## Standard Results 1

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $a^x$ | $a^x \log_e a$ |
| $\log_e x$ | $\frac{1}{x}$ |
| $\log_a x$ | $\frac{1}{x \log_e a}$ |

## Standard Results 2

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-\csc^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\csc x$ | $-\csc x \cot x$ |